Random partitions, potential of the Shapley value, and games with externalities

André Casajus[a], Yukihiko Funaki[b], Frank Huettner[c,*]

[a]HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany
[b]School of Political Science and Economics, Waseda University, Nishi-Waseda, Shinjuku-Ku, Tokyo, Japan
[c]SKK GSB, Sungkyunkwan University, 25-2, Sungkyunkwan-Ro, Jongno-gu, 110-745 Seoul, South Korea

Abstract

The Shapley value equals a player's contribution to the potential of a game. The potential is a most natural one-number summary of a game, which can be computed as the expected accumulated worth of a random partition of the players. This computation integrates the coalition formation of all players and readily extends to games with externalities. We investigate those potential functions for games with externalities that can be computed this way. It turns out that the potential that corresponds to the MPW solution introduced by Macho-Stadler et al. (2007, J. Econ. Theory 135, 339-356), is unique in the following sense. It is obtained as a the expected accumulated worth of a random partition, it generalizes the potential for games without externalities, and it induces a solution that satisfies the null player property even in the presence of externalities.

Keywords: Shapley value, partition function form, random partition, restriction operator, Ewens distribution, Chinese restaurant process, potential, externalities, null player, expected accumulated worth
2010 MSC: 91A12, *JEL:* C71, D60

*Corresponding author
Email addresses: mail@casajus.de (André Casajus), funaki@waseda.jp (Yukihiko Funaki), mail@frankhuettner.de (Frank Huettner)
URL: www.casajus.de (André Casajus), yfunaki.blogspot.com (Yukihiko Funaki), www.frankhuettner.de (Frank Huettner)

1. Introduction

A significant part of the knowledge regarding the Shapley value is based on understanding how it behaves in relation to subgames. The reference to subgames might appear rather subtle as in the carrier axiom by Shapley (1953), or clearly emphasized as in other major characterizations, e.g., by Sobolev (1975), Myerson (1980), and Hart and Mas-Colell (1989). The latter concretize the idea that a player's payoff can be obtained as a contribution to the *game*, i.e., to a one-number summary of a game. They call this one-number summary a *potential* and obtain uniqueness of both the potential as well as the Shapley value being a player's contribution to the potential. Hart and Mas-Colell (1989) further emphasize the importance of the fact that the potential can be computed in an intuitive way as the expected normalized worth in a game. Whereas this computation considers coalitions in isolation, Casajus (2014) demonstrates that the potential can alternatively be computed as the expected accumulated worth of a random partition of *all* players. This perspective, which evaluates a coalition's worth in the context of the coalition formation of the other players, is particularly striking if we want to extend these ideas to a setup that includes externalities.

Cooperative games with e*x*ternalities (henceforth *TUX games*, also known as games in partition function form) emerge as more general approach to cooperative games with transferable utility (henceforth *TU games*). The presence of externalities means that the worth generated by a coalition depends on the coalitions formed by the other players (Thrall and Lucas, 1963). In search for a generalization of the Shapley value (Shapley, 1953), several solutions concepts for TUX games have been suggested. We highlight Dutta et al. (2010), who study subgames and the potential approach for TUX games and address the following challenge. In the presence of externalities, one cannot simply read off a subgame from the original game as it is the case for TU games. Dutta et al. (2010) argue that there are many ways to create subgames of a TUX game. This motivates them to introduce the notion of restriction operators. A restriction operator specifies how subgames are derived from TUX games. A restriction operator is path independent if it does not matter in which order the players are removed from a TUX game. Dutta et al. (2010) show that once a path independent restriction operator has been selected, then there exists a unique potential for TUX games and a unique corresponding solution for TUX games that gives a player the contribution to this potential.

In this paper, we connect these insights by asking wether there is a one-number summary of each TUX game, such that (i) it is obtained as the expected accumulated worth of some random partition of the players as in Casajus (2014); (ii) it is induced by a plausible restriction operator, which connects TUX games with their subgames as in Dutta et al. (2010); and (iii) a player's contribution to this number provides a generalization of the Shapley value to TUX games, which inherits the basic properties of the Shapley value (Shapley, 1953). The answer is affirmative and yields a new foundation of the MPW solution introduced by Macho-Stadler et al. (2007).[1]

[1]This solution is also known as stochastic Shapley value (Skibski et al., 2018) or, somewhat imprecisely, as *the* average Shapley value.

The remainder of this paper is structured as follows. In Section 2, we provide the basic notation and revisit the relevant results. First, we focus on TU games and the Shapley value, which is the contribution to the potential of a TU game, which in turn can be obtained as the expected accumulated worth of a random partition. Then, we provide the basic notation and revisit the relevant results on TUX games, in particular the approach to subgames via restriction operators by Dutta et al. (2010).

In Section 3, we provide our results. First, we revisit the result of Casajus (2014), who has demonstrated that the Ewens distribution with mutation rate $\theta = 1$ (Ewens, 1972; Karlin and McGregor, 1972) generates the potential for TU games as the expected accumulated worth of a random partition of the player set. We demonstrate that there are further such random partitions and investigate their properties.

We then obtain our first main result. Given a positive random partition that generates the potential for TU games, we can construct a restriction operator such that the expected accumulated worth of the random partition corresponds to the potential for TUX games. Moreover, our construction is unique under some mild assumptions on the restriction operator.

Our second main result comes from the study of those restriction operators that we constructed and their induced solutions for TUX games. We find that among these solutions, only the MPW solution satisfies the null player property (or monotonicity) for TUX games. There, we also uncover the specific restriction operator that naturally leads to the MPW solution. This restriction operator is reminiscent of the Chinese restaurant process, which is known to yield the Ewens distributions (Aldous, 1985, 11.19; Pitman, 2006, Equation 3.3).

We then discuss the relations between our approach and the solutions proposed in the literature. Thereafter, we conclude with some remarks on the following points. The restriction operator that yields the MPW solution exhibits a surprising property, namely that it may convert a null player into a non-null player when another player is removed. Yet, a previously null player still obtains zero according to the MPW solution in the subgame. Moreover, we discuss the relationship between the condition that a random partition generates the potential for TU games and the well-known conditional independence property. Finally, we outline how our work may help to provide the basis for future research. The appendix contains all the proofs.

2. Preliminaries

We introduce notation and results that are needed to understand our results in several parts. First, we stay within the world of TU games. Then, we include externalities. The appendix includes additional notation that is needed for the proofs.

2.1. The Shapley value is a player's contribution to the expected accumulated worth of a partition

Let $\mathbf{U}$ be a finite set of players, the **universe of players**. Throughout the paper, the cardinalities of coalitions $N, S, T, B, C \subseteq \mathbf{U}$ are denoted by n, s, t, b, and c, respectively. A **partition** of $N \subseteq \mathbf{U}$ is a collection of non-empty subsets of N such that any two of them are disjoint and such their union is N. The **set of partitions** of N is denoted by $\Pi(N)$. Note that $\Pi(\emptyset) = \{\emptyset\}$. A **random partition** for $\mathbf{U}$ is a family of probability distributions $p = (p_N)_{N\subseteq \mathbf{U}}$ over partitions, i.e., p_N is a probability distribution over $\Pi(N)$. For $\pi \in \Pi(N \setminus \{i\})$, **adding the player** $i \in N$ to block $B \in \pi$ is denoted by $\pi_{+i\rightsquigarrow B} \in \Pi(N)$,

$$\pi_{+i\rightsquigarrow B} := (\pi \setminus \{B\}) \cup \{B \cup \{i\}\};$$

adding player i as a singleton is denoted by $\pi_{+i\rightsquigarrow\emptyset} \in \Pi(N)$, i.e., $\pi_{+i\rightsquigarrow\emptyset} := \pi \cup \{\{i\}\}$.

A cooperative game with transferable utility, henceforth **TU game** (also known as a game in characteristic function form), for a player set $N \subseteq \mathbf{U}$ is given by its **characteristic function** $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, which assigns a **worth** to each coalition $S \subseteq N$. Let $\mathbb{V}(N)$ denote the set of all TU games for N and let $\mathbb{V}$ denote the set of all TU games. For $v \in \mathbb{V}(N)$ and $i \in N$, the **subgame** v_{-i} on the player set $N \setminus \{i\}$ is simply the **restriction** of v and given by $v_{-i}(S) = v(S)$ for all $S \subseteq N \setminus \{i\}$. A one-point **solution for TU games** is an operator φ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^N$ to every TU game $v \in \mathbb{V}(N)$ and player set $N \subseteq \mathbf{U}$.

The Shapley value (Shapley, 1953) probably is the most eminent one-point solution concept for TU games. Besides its original axiomatic foundation by Shapley himself, alternative foundations of different types have been suggested later on. Important direct axiomatic characterizations are due to Myerson (1980) and Young (1985). The **Shapley value**, Sh, is given by

$$\mathrm{Sh}_i(v) := \sum_{S\subseteq N\setminus\{i\}} \frac{s!\,(n-s-1)!}{n!} \left(v(S\cup\{i\}) - v(S)\right) \tag{1}$$

for all $N \subseteq \mathbf{U}$, $v \in \mathbb{V}(N)$, and $i \in N$.

Hart and Mas-Colell (1989) suggest an indirect characterization of the Shapley value by means of a potential (function) for TU games. A **potential** is a mapping $\mathrm{Pot} : \mathbb{V} \to \mathbb{R}$ that satisfies the following two properties. Zero-normalization: For $v_\emptyset \in \mathbb{V}(\emptyset)$, we have $\mathrm{Pot}(v_\emptyset) = 0$, i.e., the unique game for the empty player set has a zero potential. Efficiency: For all $N \subseteq \mathbf{U}$, $v \in \mathbb{V}(N)$,we have

$$\sum_{i\in N} \left[\mathrm{Pot}(v) - \mathrm{Pot}(v_{-i})\right] = v(N). \tag{2}$$

It turns out that there exists a unique potential and that a player's contribution to the potential equals this player's Shapley payoff.[2]

Theorem 1 (Hart and Mas-Colell, 1989). *There exists a unique potential for TU games and we have*

$$\mathrm{Sh}_i(v) = \mathrm{Pot}(v) - \mathrm{Pot}(v_{-i}) \qquad \textit{for all } N \subseteq \mathbf{U},\ v \in \mathbb{V}(N),\ \textit{and } i \in N. \tag{3}$$

[2] Calvo and Santos (1997) and Ortmann (1998) generalize the notion of a potential to non-efficient solutions.

Hart and Mas-Colell (1989) establishes that the potential can be calculated as the expected accumulated worth of a random partition.

Theorem 2 (Casajus, 2014). *For all $N \subseteq \mathbf{U},\ v \in \mathbb{V}(N)$, we have*

$$\operatorname{Pot}(v) = \sum_{\pi \in \Pi(N)} p_N^\star(\pi) \sum_{B \in \pi} v(B), \tag{4}$$

where the random partition $p^\star$ is given by

$$p_N^\star(\pi) := \frac{\prod_{B \in \pi} (b-1)!}{n!} \qquad \text{for all } \pi \in \Pi(N). \tag{5}$$

The random partition $p^\star$ is known as the **Ewens distribution with mutation rate $\theta = 1$** (Ewens, 1972; Karlin and McGregor, 1972), which plays a central role in the literature on random partitions (Crane, 2016).[3]

As a consequence, we can understand the Shapley value as a player's contribution to the expected accumulated worth of a partition. The Shapley value, Sh, is given by

$$\operatorname{Sh}_i(v) = \sum_{\pi \in \Pi(N)} p_N^\star(\pi) \sum_{B \in \pi} v(B) - \sum_{\tau \in \Pi(N \setminus \{i\})} p_{N \setminus \{i\}}^\star(\tau) \sum_{C \in \tau} v_{-i}(C) \tag{6}$$

for all $N \subseteq \mathbf{U}$, $v \in \mathbb{V}(N)$, and $i \in N$. Note that the right-hand-sides of (3) and (6) refer to games with varying player sets. In this sense, the Shapley value emerges as a player's *contribution to a game* and not as the weighted contribution to coalitions. Operating on varying player sets is also useful for implementations and bargaining foundations of the value, which often rest on the idea that some player's productivity is bought out by other players, stepwise reducing the player set (prominent examples include Gul (1989), Stole and Zwiebel (1996), Macho-Stadler et al. (2007), McQuillin and Sugden (2016), Brügemann et al. (2018)).

Thinking of a game in terms of the expected accumulated worth of a partition is particularly striking if we keep in mind that the value of a coalition should be understood in context of the coalition structure of outsiders. This is important in the context of externalities, i.e., if the value of a coalition depends on the coalitions formed by the other players. Naturally, we arrive at the question: *Can (6) be generalized towards a setup that takes externalities into account?*

Of course, intuitive formulas for the potential as in (4) are of interest by itself. For example, Hart and Mas-Colell (1989, Proposition 2.4) emphasize that the per-capita potential of a game is the expected per-capita worth of a standard random coalition. First, any coalition size is chosen with the same probability. Second, any coalition of a given size is chosen with

[3] The one-parametric family of Ewens distributions p^θ, $\theta > 0$ is given by $p_N^\theta(\pi) := \frac{\Gamma(\theta)\theta^{|\pi|}}{\Gamma(\theta+n)} \prod_{B \in \pi} (b-1)!$ for $N \subseteq \mathbf{U},\ \pi \in \Pi(N)$, where Γ denotes the Gamma function. For all $\theta > 0$, $N \subseteq \mathbf{U}$, and $i, j \in N$, $i \neq j$, we have $\sum_{\pi \in \Pi(N) : \pi(i) = \pi(j)} p_N^\theta(\pi) = \frac{1}{1+\theta}$. Note that a mutation rate of 1 corresponds to a probability of $1/2$ of two arbitrary players belonging to the same block of a partition.

the same probability. That is, we have

$$\frac{\mathrm{Pot}\,(v)}{n}=\sum_{S\subseteq N:S\neq\emptyset}\frac{1}{n}\frac{s!\,(n-s)!}{n!}\frac{v\,(S)}{s}\qquad\text{for all } N\subseteq\mathbf{U},\ v\in\mathbb{V}\,(N)\,. \tag{7}$$

They conclude that this formula indicates that "... the potential provides a most natural one-number summary of the game" (Hart and Mas-Colell, 1989, p. 592). Likewise, a potential for games with externalities that exhibits some intuitive formula shall be appreciated.

Finally, we shall notice that the random partition $p^\star$ is the consequence of a stochastic coalition formation process known as the "*(uniform) Chinese restaurant process*" (Aldous, 1985, 11.19; Pitman, 2006, Equation 3.3). It can be described as follows. Consider a restaurant with n round tables, each seating up to n persons. The players arrive at the restaurant in some order. The first player takes a seat at one of the tables. Any following player takes seat at an empty table or joins any of the already present players with the *same* probability. This motivates the use of "*uniform*" above. This process induces a random partition where each table will be understood as a block of a partition. Concretely, after n players have entered the room and taken a seat, each occupied table will be understood as a block of $\pi\in\Pi\,(N)$ and π has the probability $p_N^\star\,(\pi)$. Hence, the probability distributions $p_N^\star$ and $p_{N\setminus\{i\}}^\star$ are connected by

$$p_N^\star\,(\pi_{+i\rightsquigarrow\emptyset})=\frac{1}{n}p_{N\setminus\{i\}}^\star\,(\pi)\qquad\text{and}\qquad p_N^\star\,(\pi_{+i\rightsquigarrow B})=\frac{b}{n}p_{N\setminus\{i\}}^\star\,(\pi)\,. \tag{8}$$

Insertion of (8) into (6) yields a formula of the Shapley value as the expected marginal contribution to coalitions over all partitions. The Shapley value, Sh, is given by

$$\mathrm{Sh}_i\,(v)=\sum_{\pi\in\Pi(N\setminus\{i\})}p_{N\setminus\{i\}}^\star\,(\pi)\left(\frac{1}{n}\left(v\left(\{i\}\right)-v\left(\{\emptyset\}\right)\right)+\sum_{S\in\pi}\frac{s}{n}\left(v\left(S\cup\{i\}\right)-v\left(S\right)\right)\right) \tag{9}$$

for all $N\subseteq\mathbf{U}$, $v\in\mathbb{V}\,(N)$, and $i\in N$. In words, the Shapley value is a player's expected marginal contribution to a table when entering the Chinese restaurant process last. This might, for instance, be helpful for computations of the Shapley value based on Monte Carlo sampling from random partitions.[4]

In this paper, however, we primarily focus on formulas that investigate a player's *contribution to a game* as in (6), rather than on formulas that exhibit a player's *contribution to coalition* as for instance in (9). For the latter, we refer to (Skibski et al., 2018) and Skibski and Michalak (2019), who study games with externalities and solutions thereof in the context of stochastic coalition formation processes.

[4]Standard Monte Carlo algorithms to compute the Shapley value rely on sampling from rank orders, e.g., Castro et al. (2009); however, exploiting particular formulas has shown to be more efficient for some games, e.g., Michalak et al. (2013) or Lundberg et al. (2020).

2.2. Games with externalities and subgames

A TU game with externalities, henceforth **TUX game** (also known as **game in partition function form**), for a player set $N \subseteq \mathbf{U}$ is given by its **partition function** $w : \mathcal{E}(N) \to \mathbb{R}^N$, where $\mathcal{E}(N)$ denotes the set of **embedded coalitions** (S,π) for N given by

$$\mathcal{E}(N) := \{(S,\pi) \mid S \subseteq N \text{ and } \pi \in \Pi(N \setminus S)\},$$

and with $w(\emptyset,\pi) = 0$ for all $\pi \in \Pi(N)$. We denote the set of all TUX games for a player set N by $\mathbb{W}(N)$ and the set of all TUX games by $\mathbb{W}$.

If there are no externalities, that is, if $w(S,\pi) = w(S,\tau)$ for all $S \subseteq N$ and all $\pi,\tau \in \Pi(N \setminus S)$, then the game w effectively is just a TU game. In this sense TU games are special TUX games, $\mathbb{V} \subseteq \mathbb{W}$. An example of a game without externalities is the **null game** $\mathbf{0}^N \in \mathbb{W}(N)$, $N \subseteq \mathbf{U}$, given by $\mathbf{0}^N(S,\pi) = 0$ for all $(S,\pi) \in \mathcal{E}(N)$.

A one-point **solution for TUX games** is an operator φ that assigns a payoff vector $\varphi(w) \in \mathbb{R}^N$ to every TUX game $w \in \mathbb{W}(N)$ and player set $N \subseteq \mathbf{U}$. Whereas there is no eminent solution for TUX games, the literature provides numerous **generalizations of the Shapley value**, i.e., solutions for TUX games that coincide with the Shapley value on TU games, $\varphi(v) = \mathrm{Sh}(v)$ for all $v \in \mathbb{V}(N) \subseteq \mathbb{W}(N)$, $N \subseteq \mathbf{U}$. Examples include Myerson (1977),Bolger (1989), Albizuri et al. (2005), Pham Do and Norde (2007), McQuillin (2009), and Dutta et al. (2010); see Kóczy (2018) for a survey. We highlight the solution introduced by (Macho-Stadler et al., 2007), which is obtained as the Shapley value of some average TU game, i.e.,

$$\mathrm{MPW}(w) := \mathrm{Sh}(\bar{v}_w^\star) \qquad \text{for all } N \subseteq \mathbf{U},\ w \in \mathbb{W}(N), \tag{10}$$

where the average TU game $\bar{v}_w^\star \in \mathbb{V}(N)$, in which each coalition S gets the expected value of $w(S,\pi)$ over all partitions $\pi \in \Pi(N \setminus S)$ using the random partition $p^\star$, is given by

$$\bar{v}_w^\star(S) := \sum_{\pi \in \Pi(N\setminus S)} p_{N\setminus S}^\star(\pi)\, w(S,\pi) \qquad \text{for all } S \subseteq N. \tag{11}$$

Both the potential and the understanding of the Shapley value as a player's contribution to the expected accumulated worth of a partition rely on the notion of subgames, which result from reducing the player set. For TU games, there is an obvious way to define subgames. In contrast, the notion of a subgame is less obvious for TUX games, since we cannot simply read it off the original game. When player i is removed from the TUX game w, we have to specify the worth of each embedded coalition "$w_{-i}(S,\pi)$" in the TUX game w_{-i} without player i. Other than for TU games, it is unclear how "$w_{-i}(S,\pi)$" is obtained from w.

Dutta et al. (2010) introduce the concept of a **restriction operator** r that specifies for each TUX game $w \in \mathbb{W}(N)$, $N \subseteq \mathbf{U}$, and player $i \in N$ a "subgame" $w_{-i}^r \in \mathbb{W}(N \setminus \{i\})$, where the worth $w_{-i}^r(S,\pi)$ in the subgame only depends on the worths of those embedded coalitions in the original game that are obtained when player i is added to a block in π or stays alone. That is, a restriction operator satisfies the following property.

Restriction, RES. For all $N \subseteq \mathbf{U}$, $w, w' \in \mathbb{W}(N)$, $i \in N$, and $(S,\pi) \in \mathcal{E}(N \setminus \{i\})$ such that $w(S,\pi_{+i\rightsquigarrow B}) = w'(S,\pi_{+i\rightsquigarrow B})$ for all $B \in \pi \cup \{\emptyset\}$, we have $w_{-i}^r(S,\pi) = (w')_{-i}^r(S,\pi)$.

Another crucial property of restriction operators is called path independence. A restriction operator is path independent if the order in which the players are removed is irrelevant.

Path independence, PI. For all $N \subseteq \mathbf{U}$, $w \in \mathbb{W}(N)$, and $i, j \in N$, $i \neq j$, we have $(w^r_{-i})^r_{-j} = (w^r_{-j})^r_{-i}$.

Note that we can refer to subgames obtained by the elimination of multiple players whenever r is path independent. That is, the subgames w^r_{-T}, $T \subseteq N$, are well-defined for path independent restriction operators.

With the ability to remove players from a game, Dutta et al. (2010) can generalize the notion of potential functions to TUX games. Given a path independent restriction operator r, define the **r-potential** by replacing v_{-i} in (2) with w^r_{-i}. That is, an r-potential is a mapping $\mathrm{Pot}^r : \mathbb{W} \to \mathbb{R}$ that satisfies the following two properties: Zero-normalization: For $\mathbf{0}^{\emptyset} \in \mathbb{W}(\emptyset)$, we have $\mathrm{Pot}^r\left(\mathbf{0}^{\emptyset}\right) = 0$, i.e., the unique game for the empty player set has a zero potential. Efficiency: For all $N \subseteq \mathbf{U}$, $w \in \mathbb{W}(N)$, we have

$$\sum_{i \in N} \left[\mathrm{Pot}^r(w) - \mathrm{Pot}^r\left(w^r_{-i}\right)\right] = w(N, \emptyset). \tag{12}$$

This recursively determines a well-defined and unique potential for each path independent restriction operator r.

Similarly to the Shapley value, which is the contribution of a player to the potential of a TU game, each path independent restriction operator induces a unique solution for TUX games, the **r-Shapley value**, which is the contribution of a player to the r-potential of a TUX game. The connection between the r-Shapley value and the Shapley value is via auxiliary TU games $v^r_w \in \mathbb{V}(N)$:

1. For a given TUX game $w \in \mathbb{W}(N)$, $N \subseteq \mathbf{U}$, we first compute the auxiliary TU game $v^r_w \in \mathbb{V}(N)$, in which each coalition S generates the worth generated by the grand coalition if we remove all the other players $N \setminus S$ from the game. That is, the auxiliary TU game is defined by

$$v^r_w(S) := w^r_{-N \setminus S}(S, \emptyset) \qquad \text{for all } S \subseteq N. \tag{13}$$

2. Second, we apply the Shapley value to this auxiliary TU game in order to obtain the r-Shapley value,

$$\mathrm{Sh}^r(w) := \mathrm{Sh}(v^r_w) \qquad \text{for all } w \in \mathbb{W}(N),\ N \subseteq \mathbf{U}. \tag{14}$$

Dutta et al. (2010) find that the r-potential of the TUX game w is just the potential of the auxiliary TU game v^r_w and that the r-Shapley value is a player's contribution to the r-potential of a TUX game.

Theorem 3 (Dutta et al., 2010). *For every path independent restriction operator r, there exists a unique r-potential Pot^r. It is given by*

$$\mathrm{Pot}^r(w) = \mathrm{Pot}(v^r_w) \qquad \textit{for all } N \subseteq \mathbf{U},\ w \in \mathbb{W}(N), \tag{15}$$

where v_w^r is defined in (13). Moreover, we have

$$\mathrm{Sh}_i^r\left(w\right) = \mathrm{Pot}^r\left(w\right) - \mathrm{Pot}^r\left(w_{-i}^r\right) \qquad \textit{for all } N \subseteq \mathbf{U},\ w \in \mathbb{W}\left(N\right),\ \textit{and } i \in N. \tag{16}$$

Equipped with these insights, we can now proceed to our research questions.

3. Results

In this section, we investigate if it is possible to summarize each TUX game by a single number such that this number is obtained as the r-potential of a TUX game for some restriction operator r, and such that this number is also obtained as the **expected accumulated worth of some random partition** p, given by

$$\mathrm{E}_p\left(w\right) := \sum_{\pi \in \Pi(N)} p_N\left(\pi\right) \sum_{S \in \pi} w\left(S, \pi \setminus \{S\}\right) \qquad \text{for all } N \subseteq \mathbf{U},\ w \in \mathbb{W}\left(N\right). \tag{17}$$

Moreover, we want this one-number summary to equal the potential for TU games if there are no externalities. Hence, we are interested in random partitions that satisfy the following property.

Potential generating, GEN. For all TU games $v \in \mathbb{V}\left(N\right) \subseteq \mathbb{W}\left(N\right)$ and player sets $N \subseteq \mathbf{U}$, we have $\mathrm{E}_p\left(v\right) = \mathrm{Pot}\left(v\right)$, where the potential for TU games $\mathrm{Pot}\left(v\right)$ defined in (2).

The answer is affirmative and will be constructive, i.e., we can provide a formula for the respective restriction operators. This leads to our final question: Do the induced r-Shapley values preserve standard properties of the Shapley value. As it turns out, this is only the case for the MPW solution introduced by Macho-Stadler et al. (2007).

We structure our journey as follows. To begin with, we shed some light on random partitions that generate the potential and motivate further plausible conditions on random partitions. Thereafter, we describe the structure of restriction operators r^p, such that $\mathrm{Pot}^{r^p} = \mathrm{E}_p$ for some random partition p that generates the potential for TU games (**GEN**). Finally, we identify the MPW solution as the only resulting r^p-Shapley value that satisfies the null player property in presence of externalities.

3.1. Random partitions partition that generate the potential for TU games

For games with more than four players, the Ewens distribution $p^\star$ is not the only random partition that generates the potential for TU games (**GEN**). More precisely, the following proposition describes the characteristics of potential-generating random partitions.

Proposition 4. *The following statements are equivalent:*

*(i) The random partition p generates the potential for TU games (**GEN**).*

(ii) The random partition p satisfies

$$\sum_{\pi \in \Pi(N): T \in \pi} p_N\left(\pi\right) = \frac{(n-t)!\,(t-1)!}{n!} \qquad \textit{for all } N \subseteq \mathbf{U},\ T \subseteq N,\ T \neq \emptyset. \tag{18}$$

(iii) The random partition p satisfies

$$\sum_{\pi\in\Pi((N\setminus\{i\})\setminus S)} p_{N\setminus\{i\}}\left(\{S\}\cup\pi\right) = \frac{n}{n-s}\sum_{\pi\in\Pi((N\setminus\{i\})\setminus S)}\ \sum_{B\in\pi\cup\{\emptyset\}} p_N\left(\{S\}\cup\pi_{+i\rightsquigarrow B}\right) \tag{19}$$

for all $N\subseteq\mathbf{U},\ i\in N$, and $S\subseteq N\setminus\{i\}\,,\ S\neq\emptyset$.

Whereas condition (ii) gives an explicit expression for the probability of a particular coalition to appear in a randomly chosen partition, condition (iii) offers a recursive relation. The latter will be useful for the construction of a restriction operator that "behaves well" on TU games. Note that these conditions already determine the probabilities $p_N\left(\{N\}\right)$ and $p_N\left(\{N\setminus\{i\}\,,\{i\}\}\right)$. Moreover, for games with up to three players, the random partition p needs to coincide with the Ewens distribution $p^\star$.

Corollary 5. *(i) For every random partition p that generates the potential for TU games (**GEN**), we have $p_N\left(\{N\}\right)=\frac{1}{n}$ and $p_N\left(\{N\setminus\{i\}\,,\{i\}\}\right)=\frac{1}{n(n-1)}$ for all $N\subseteq\mathbf{U}$ and $i\in N$.*

(ii) For all $N\subseteq\mathbf{U}$ such that $n\leq 3$, we have $p_N\left(\pi\right)=p^\star_N\left(\pi\right)$ for all $\pi\in\Pi\left(N\right)$, where $p^\star$ is given in (5).

In games with more than three players, we obtain additional degrees of freedom. For instance, every random partition p^c, $-\frac{1}{24}\leq c\leq\frac{1}{8}$, that assigns the following probabilities to partitions of four players, $\mathbf{U}=\{i,j,k,\ell\}$, generates the potential for TU games:

$$p^c_4\left(\pi\right)=\begin{cases} p^\star_4\left(\pi\right)+\frac{1}{3}c, & \pi=\{\{i,j\}\,,\{k,\ell\}\}\,,\\ p^\star_4\left(\pi\right)-\frac{1}{3}c, & \pi=\{\{i,j\}\,,\{k\}\,,\{\ell\}\}\,,\\ p^\star_4\left(\pi\right)+c, & \pi=\{\{i\}\,,\{j\}\,,\{k\}\,,\{\ell\}\}\,,\\ p^\star_4\left(\pi\right), & \text{otherwise.}\end{cases} \tag{20}$$

For details on how to extend (20) to larger games, we refer to the appendix.

Note that setting $c=\frac{1}{8}$ and $c=-\frac{1}{24}$ in (20) gives random partitions that assign zero probability to partitions of types $\{\{i,j\}\,,\{k\}\,,\{\ell\}\}$ and $\{\{i\}\,,\{j\}\,,\{k\}\,,\{\ell\}\}$, respectively. In what follows, we might not be interested in such random partitions. Instead, we usually want that the expected accumulated worth of a random partition makes use of all the information of a game and we shall assume the following positivity condition on random partitions.

Positivity, POS. For all $N\subseteq\mathbf{U}$ and $\pi\in\Pi\left(N\right)$, we have $p_N\left(\pi\right)>0$.

To rule out further oddities, we shall restrict our focus to restriction operators that map null games to null games. Specifically, we shall impose the following condition on restriction operators.

Preservation of null games, PNG. For all $N\subseteq\mathbf{U}$ and $i\in N$, we have $\left(\mathbf{0}^N\right)^r_{-i}=\mathbf{0}^{N\setminus\{i\}}$.

3.2. Potentials for TUX games equaling the expected accumulated worth of a partition

We are ready to develop our first main result. It describes the structure of those restriction operators that induce a potential that both generalizes the potential for TU games and is given as the expected accumulated worth of a random partition. Suppose we are given some positive random partition p that generates the potential for TU games. In search of a suitable restriction operator r^p, we need to preserve the natural restriction on TU games, i.e., $v_{-i}^{r^p}(S,\pi) = v(S,\pi_{+i\rightsquigarrow B})$ for all $v \in \mathbb{V}(N) \subseteq \mathbb{W}(N)$, $(S,\pi) \in \mathcal{E}(N\setminus\{i\})$, and $B \in \pi \cup \{\emptyset\}$. By Proposition 4 (iii), we then have

$$\sum_{\tau\in\Pi((N\setminus\{i\})\setminus S)} p_{N\setminus\{i\}}(\{S\}\cup\pi)\, v_{-i}^{r^p}(S,\pi)$$
$$= \frac{n}{n-s} \sum_{\tau\in\Pi((N\setminus\{i\})\setminus S)} \sum_{B\in\pi\cup\{\emptyset\}} p_N(\{S\}\cup\pi_{+i\rightsquigarrow B})\, v(S,\pi_{+i\rightsquigarrow B}).$$

This indicates that the following restriction operator works well for TU games $\mathbb{V}(N) \subseteq \mathbb{W}(N)$.

Definition 6. *Let p be a positive (**POS**) random partition that generates the potential for TU games (**GEN**). We define the restriction operator r^p by*

$$w_{-i}^{r^p}(S,\pi) := \frac{n}{n-s} \sum_{B\in\pi\cup\{\emptyset\}} \frac{p_N(\{S\}\cup\pi_{+i\rightsquigarrow B})}{p_{N\setminus\{i\}}(\{S\}\cup\pi)} w(S,\pi_{+i\rightsquigarrow B}) \tag{21}$$

for all $N\subseteq \mathbf{U}$, $w\in\mathbb{W}(N)$, $i\in N$, and $(S,\pi)\in\mathcal{E}(N\setminus\{i\})$.

It remains to be clarified whether this restriction operator also has plausible properties for games with externalities. This is the content of the following theorem, which further emphasizes the *uniqueness* of this restriction operator.

Theorem 7. *Let p be a positive random partition (**POS**) that generates the potential for TU games (**GEN**). A restriction operator r is path independent (**PI**), preserves null games (**PNG**), and satisfies $\mathrm{Pot}^r = \mathrm{E}_p$, i.e., its induced r-potential defined in (15) coincides with the expected accumulated worth of the random partition p given in (17), if and only if $r = r^p$.*

Theorem 7 is useful for multiple reasons. It clarifies that whenever a given positive random partition p generates the potential for TU games as the expected accumulated worth of the random partition, then this can be extended to a potential for TUX games in a unique plausible way. This is achieved by using the restriction operator r^p given in (21), which yields the auxiliary games $v_w^{r^p}$ as defined in (13), thereby yielding the potential Pot^{r^p} according to (15).

Moreover, Theorem 7 is constructive in the sense that it yields a tractable formula for the restriction operator r^p. This has immediate consequences, such as a simple formula for the induced r^p-Shapley value, which by definition is a player's contribution to the r^p-potential of a TUX game.

Corollary 8. *(i) Let p be a positive random partition (**POS**) that generates the potential for TU games (**GEN**) and denote the corresponding restriction operator according to (21) by r^p. The r^p-Shapley value Sh^{r^p} as defined by (14) is given by*

$$\mathrm{Sh}_i^{r^p}(w) = \sum_{(T,\tau)\in\mathcal{E}(N\setminus\{i\})} \Bigg(p_N\left(\{T\cup\{i\}\}\cup\tau\right) w\left(T\cup\{i\},\tau\right) - \frac{t}{n-t}\sum_{B\in\tau\cup\{\emptyset\}} p_N\left(\{T\}\cup\tau_{+i\rightsquigarrow B}\right) w\left(T,\tau_{+i\rightsquigarrow B}\right)\Bigg) \tag{22}$$

for all $N\subseteq\mathbf{U},\ w\in\mathbb{W}(N)\,,$ and $i\in N$.

(ii) Moreover, for TU games, the r^p-Shapley value Sh^{r^p} coincides with the Shapley value, i.e., $\mathrm{Sh}^{r^p}(v)=\mathrm{Sh}(v)$ for all $N\subseteq\mathbf{U}$, $v\in\mathbb{V}(N)\subseteq\mathbb{W}(N)$.

The corollary reconfirms that the r^p-Shapley value generalizes the Shapley value for suitable random partitions. Moreover, it provides a formula that allows us to investigate its properties.

3.3. The only r^p-Shapley value satisfying the null player player property is the MPW solution

So far we have motivated the r^p-Shapley value, which emerges if we want to generalize the potential from TU games to TUX games in a way that preserves its interpretation of being an expected accumulated worth of a random partition. We further identified the specific structure of such r^p-Shapley values.

From (22), it becomes apparent that the r^p-Shapley values are additive in the game and anonymous. Moreover, every r^p-Shapley value inherits efficiency from the Shapley value by definitions (14) and (13). However, it is less clear whether r^p-Shapley values satisfy standard monotonicity properties or assign zero payoff to null players. Indeed, the latter is at the heart of our second main result. We prepare it by investigating the obvious question of what we obtain if we apply Theorem 7 to the Ewens distribution $p^\star$ given in (5).

Lemma 9. *For $p=p^\star$, the restriction operator $r^\star := r^{p^\star}$ is given by*

$$w_{-i}^{r^\star}(S,\pi) = \frac{1}{n-s} w\left(S,\pi_{+i\rightsquigarrow\emptyset}\right) + \sum_{B\in\pi}\frac{b}{n-s} w\left(S,\pi_{+i\rightsquigarrow B}\right) \tag{23}$$

for all $N\subseteq\mathbf{U}$, $w\in\mathbb{W}(N)$, $i\in N$, and $(S,\pi)\in\mathcal{E}(N\setminus\{i\})$. Moreover, the $r^\star$-Shapley value coincides with the MPW solution, $\mathrm{Sh}^{r^\star}=\mathrm{MPW}$.

Lemma 9 highlights that $p^\star$ yields an interesting restriction and a very familiar generalization of the Shapley value. Moreover, the restriction operator lends itself to an intuitive interpretation reminiscent of the Chinese restaurant process. The worth of an embedded coalition after player i has left the game computes as if this player would take a seat next to any outside player equally likely or stays alone.

From (10) and (23) it follows that the MPW solution assigns a zero payoff to null players. Specifically, player $i \in N$ is a **null player** in TUX game $w \in \mathbb{W}(N)$ with player set $N \subseteq \mathbf{U}$ if player i's position in the game does not influence the worth of any embedded coalition, i.e., if $w(S \cup \{i\}, \pi) = w(S, \pi_{+i \rightsquigarrow B})$ for all $(S, \pi) \in \mathcal{E}(N \setminus \{i\})$ and $B \in \pi \cup \{\emptyset\}$.[5] The MPW solution satisfies the following property.[6]

Null player, NP. For all $N \subseteq \mathbf{U}$, $w \in \mathbb{W}(N)$, and $i \in N$ such that player i is a null player in w, we have $\varphi_i(w) = 0$.

Notice that the null player property imposes the standard null player property on TU games $\mathbb{V} \subseteq \mathbb{W}$.[7] Indeed, since every r^p-Shapley value coincides with the Shapley value on TU games, every r^p-Shapley value also satisfies the null player property for TU games. However, the next theorem clarifies that only one r^p-Shapley value satisfies the null player property also in the presence of externalities, the MPW solution.

Theorem 10. *Let p be a random partition and denote the corresponding r^p-Shapley value defined in (22) by Sh^{r^p}. The r^p-Shapley value Sh^{r^p} satisfies the null player property (**NP**) if and only if $p = p^\star$, i.e., if and only if $\mathrm{Sh}^{r^p} = \mathrm{MPW}$.*

In summary, the MPW solution emerges as the unique solution for TUX games if we demand the following from a solution φ:

1. the TUX solution φ is an r-Shapley value for *some* path independent (**PI**) restriction operator r that preserves null games (**PNG**);
2. its corresponding r-potential coincides with the expected accumulated worth for *some* positive (**POS**) random partition that generates the potential for TU games (**GEN**); and
3. the TUX solution φ satisfies the null player property (**NP**).

4. Discussion

In this section, we discuss the relation of our approach to the literature. de Clippel and Serrano (2008) put forth another important property for solutions for TUX games. Specifically, they argue that a player's payoff shall weakly increase with this player's marginal contributions.

Monotonicity, M. For all $N \subseteq \mathbf{U}$, $w, z \in \mathbb{V}(N)$ and $i \in N$, such that

$$w(S \cup \{i\}, \pi) - w(S, \pi_{+i \rightsquigarrow B}) \geq z(S \cup \{i\}, \pi) - z(S, \pi_{+i \rightsquigarrow B})$$

for all $(S, \pi) \in \mathcal{E}(N \setminus \{i\})$ and $B \in \pi \cup \{\emptyset\}$, we have $\varphi_i(w) \geq \varphi_i(z)$.

[5] Dutta et al. (2010) call (our) null players "dummy players of type 1". Bolger (1989) calls them "dummies", Macho-Stadler et al. (2007) "dummy players", and de Clippel and Serrano (2008) "null players in the strong sense".

[6] Macho-Stadler et al. (2007) call this property the "dummy player axiom".

[7] A solution φ satisfies the null player property for TU games if for all $v \in \mathbb{V}(N) \subseteq \mathbb{W}(N)$, $N \in \mathcal{N}$, $i \in N$ such that $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i(v) = 0$.

It follows from (10) and (23) that the MPW solution satisfies monotonicity. Moreover, every r^p-Shapley value gives a zero payoff to every player in the null game, $\mathrm{Sh}_i^{r^p}\left(\mathbf{0}^N\right) = 0$ for all $i \in N$, $N \subseteq \mathbf{U}$. Hence, an r^p-Shapley value satisfies the null player property whenever it is monotonic. We can therefore replace the null player property by monotonicity in Theorem 10.

Corollary 11. *Let p be a random partition and denote the corresponding r^p-Shapley value defined in (22) by Sh^{r^p}. The r^p-Shapley value by Sh^{r^p} satisfies the monotonicity (**M**) if and only if $p = p^\star$, i.e., if and only if $\mathrm{Sh}^{r^p} = \mathrm{MPW}$.*

Note that formula (22) for the r^p-Shapley value Sh^{r^p} works for arbitrary random partitions for which we call it the p-Shapley value and denote it by Sh^p. Careful inspection of the proofs of Theorem 10 and Corollary 11 reveals that they also hold true for arbitrary random partitions, i.e., the p-Shapley values. The relation between the p-Shapley values and other TUX solutions from the literature is as follows.[8] The TUX solutions suggested by Myerson (1977), Pham Do and Norde (2007), and McQuillin (2009, Definition 4) coincide with the MPW solution for $n \leq 2$ but are not p-Shapley values for $n > 2$. The TUX solutions suggested by Bolger (1989) and Albizuri et al. (2005) coincide with the MPW solution for $n \leq 3$ but are not a p-Shapley values for $n > 3$.

5. Concluding Remarks

In this paper, we extend the stream of research on coalitional games with externalities (TUX games) and changing player sets. We start with the observation that the Shapley value is the contribution to the potential of a game, which can be computed as the expected accumulated worth of a random partition of the player set. In our quest to generalize this idea to TUX games, we need to address the challenge that there is no obvious way for creating subgames in the presence of externalities. To this end, we use the concept of restriction operators introduced by Dutta et al. (2010), which capture a plethora of possibilities to obtain subgames. Each path independent restriction operator induces a potential for TUX games. Our first main result describes those restriction operators for which the corresponding potential can be computed as an expected accumulated worth of some random partition and which generalize the potential for TU games. Each such restriction operator then induces a generalization of the Shapley value for TUX games. In particular, each such solution for TUX games gives a payoff zero to null players in the absence of externalities. However, only the MPW solution (Macho-Stadler et al., 2007) maintains this property also in the presence of externalities, which is our second main result. We further find that the notion of subgames that corresponds to the MPW solution relates to the Chinese restaurant process. The worth of an embedded coalition after a player has left computes as if this player would equally likely take a seat next to any outside player or stays alone. This reveals another connection to the Ewens distribution, which is obtained from the Chinese restaurant process.

[8]These relationships can be obtained by the straightforward but tedious calculation of the payoffs for Dirac games. Details can obtained from the authors upon request.

5.1. Null players after removing another player

Although the MPW solution is obtained as an intuitive generalization of the Shapley value that can be interpreted as a player's contribution to the expected accumulated worth of a random partition in a game, its corresponding restriction operator, i.e., its notion of how to obtain subgames, exhibits a surprising property. Suppose player j is a null player in a game and some other player i is removed from the game. It may happen that player j is no longer a null player in the subgame. Note, however, that player j's payoff according to the MPW solution remains zero. To see this more lucidly, consider the following example.

Example 12. *There are four players, $N = \{1, 2, 3, 4\}$. The partition function w is given by*

$$w(S, \pi) = \begin{cases} 1, & \textit{if } 2 \in S \textit{ and } 3 \in S, \\ 1, & \textit{if } 2 \in S,\ 3 \notin S,\ \textit{and } 4 \notin \pi(3), \\ 1, & \textit{if } 3 \in S,\ 2 \notin S,\ \textit{and } 4 \notin \pi(2), \\ 0, & \textit{otherwise.} \end{cases} \qquad \textit{for all } (S, \pi) \in \mathcal{E}(N). \tag{24}$$

In words, the position of player 1 is irrelevant. Players 2 and 3 are the productive players. If both of them are contained in S, then a worth of 1 is created. Yet, if only one of them is contained in S, then the worth created depends on the position of player 4. If player 4 is in the same block as the productive player not in S, then the worth created drops to 0.

In this game, player 1 is a null player, whereas player 4's impact originates from external effects only. Assume that player 4 leaves the game. In the resulting subgame, the worth $w^r_{-4}(\{1,3\}, \{\{2\}\})$ is determined by

$$w(\{1,3\}, \{\{2\}, \{4\}\}) = 1 \quad \text{and} \quad w(\{1,3\}, \{\{2,4\}\}) = 0,$$

and the worth $w^r_{-4}(\{3\}, \{\{1,2\}\})$ is determined by

$$w(\{3\}, \{\{1,2,4\}\}) = 0 \quad \text{and} \quad w(\{3\}, \{\{1,2\}, \{4\}\}) = 1.$$

Thus, player 1's marginal contribution $w^r_{-4}(\{1,3\}, \{\{2\}\}) - w^r_{-4}(\{3\}, \{\{1,2\}\})$ depend on these numbers. A restriction operator needs to aggregate this information and process the externalities exhibited by player 4. The restriction operator corresponding to the MPW solution $r^\star$ preserves the fact that not only players 2 and 3 have an influence on creation of worth in the original game, and make player 1 a non-null player in the subgame, resulting in the marginal contribution

$$w^{r^\star}_{-4}(\{1,3\}, \{\{2\}\}) - w^{r^\star}_{-4}(\{3\}, \{\{1,2\}\}) = \frac{1}{6}.$$

This relates to the idea that player 1's position might have had an influence on how likely player 4 "merged" with player 2. In contrast, Dutta et al. (2010) focus their analysis on restriction operators that preserve null players as such when other players are removed.

Although player 1 no longer is a null player in the subgame $w^{r^\star}_{-4}$, the MPW solution still assigns a zero payoff to this player, $\mathrm{MPW}_1\left(w^{r^\star}_{-4}\right) = 0$. This is no coincidence. As the next proposition states, null players continue to obtain a zero payoff according to the MPW solution in subgames obtained from removing other players.

Proposition 13. *For all $N \subseteq \mathbf{U}$, $w \in \mathbb{W}(N)$, $j \in N$, and $T \subseteq N \setminus \{j\}$ such that j is a null player in w, we have* $\mathrm{MPW}_j\left(w^{r^\star}_{-T}\right) = 0$.

This result rests on the fact that the average operator $\bar{v}^\star$ which is definitive of the MPW solution, the restriction operator $r^\star$, and the restriction of TU games "work together" properly, i.e., they commute.

5.2. Potential generation and conditional independence

Next, we relate the condition on a random partition p that it generates the potential for TU games (**GEN**) to another prominent condition on random partitions, which is known as conditional independence.

Conditional independence, CI (Kingman, 1978)**.** For all $N \subseteq \mathbf{U}$, $\pi \in \Pi(N)$, and $B \in \pi$, we have

$$p_N(\pi) = p_{N\setminus B}(\pi \setminus \{B\}) \sum_{\tau\in\Pi(N):B\in\tau} p_N(\tau). \tag{25}$$

With some mild additional assumptions, conditional independence is characteristic of the family of the Ewens distributions. Note that conditional independence does not imply potential generation, nor the other way around.[9] In this sense, our work complements Skibski and Michalak (2019), who highlight how conditional independence selects the MPW solution among those solutions for TUX games that can be motivated via a stochastic partition formation process. However, only one random partition satisfies both conditions.

Proposition 14. *The random partition $p^\star$ given in (5) is the unique random partition that generates the potential (**GEN**) and satisfies conditional independence (**CI**).*

5.3. Some future research paths

The research on varying player sets not only complements the research on TUX games that operates on a fixed player set (Myerson, 1977; de Clippel and Serrano, 2008; Grabisch and Funaki, 2012; Sánchez-Pérez, 2016; Skibski et al., 2018; Yang et al., 2019; Skibski and Michalak, 2019), but has multiple motivations as it also paves the way for further research. For instance, in the context of TU games, Myerson (1980) uses the removal of a player to capture a notion of fairness: player i shall benefit from player j's participation in the games as much as the other way around. Within the realm of TU games, this is as characteristic

[9]The Ewens distributions defined in Footnote 3 with mutation rates $\theta \neq 1$ are known to satisfy **CI**, but violate **GEN**, e.g., $p^{1/2}_N(\{N\}) \neq 1/n$—a contradiction to 5. The example in (20) with $c = 1/8$ satisfies **GEN**, but violates **CI**, which can be seen from $p^{1/8}_4(\{\{i,j\},\{k\},\{\ell\}\}) = 0 \neq p^\star_2(\{\{k\},\{\ell\}\}) \cdot 1/12$, a contradiction to (25) for $B = \{i,j\}$.

of the Shapley value as the admittance of a potential function (see Casajus and Huettner (2018) for a survey of equivalent properties and Kamijo and Kongo (2012) for variants that include aspects of solidarity). Analogous fairness properties can be studied for TUX games based on the intuitive restriction operator that we introduce.

Another stream of research involves the study of nonlinear generalizations of the Shapley value to TUX games. Whereas Young (1985) demonstrates that monotonicity together with efficiency and symmetry are characteristic of the Shapley value, de Clippel and Serrano (2008) emphasize that imposing these axioms in the presence of externalities does not achieve uniqueness, but even allows for nonlinear solutions. Dutta et al. (2010) demonstrate that nonlinear solutions are compatible with restriction operators and their potential approach. This invokes the question of how our formula of the potential as an expected value of a random partition needs to be adjusted to allow for such solutions as well.

Moreover, varying player sets are required for the Nash program, which aims at connecting cooperative and non-cooperative game theory. Most implementations of the Shapley value make direct or implicit use of reduced games as for example Gul (1989), Stole and Zwiebel (1996), Macho-Stadler et al. (2007), McQuillin and Sugden (2016), Brügemann et al. (2018). The present paper links the restriction operator that is characteristic of the MPW solution to the uniform Chinese restaurant process, indicating a building block for possible implementations of the MPW solution.

6. Acknowledgements

We are very grateful to a number of anonymous referees for their comments on this paper and its earlier versions. André Casajus: Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 522837108.

Appendix A. Additional notation

We denote the block of $\pi \in \Pi(N)$ that contains player $i \in N$ by $\pi(i)$. The atomistic partition is denoted by $[N] \in \Pi(N)$ and given by $[N] := \{\{i\} \mid i \in N\}$. For $\pi \in \Pi(N)$, $N \subseteq \mathbf{U}$, the elimination of the players in $T \subseteq N$ from π gives $\pi_{-T} \in \Pi(N \setminus T)$,

$$\pi_{-T} := \{B \setminus T \mid B \in \pi \text{ and } B \setminus T \neq \emptyset\}.$$

Instead of $\pi_{-\{i\}}$, we write π_{-i}.

For $\alpha \in \mathbb{R}$ and $w, w' \in \mathbb{W}(N)$, $N \subseteq \mathbf{U}$, the games $\alpha w \in \mathbb{W}(N)$ and $w + w' \in \mathbb{W}(N)$ are given by $(\alpha w)(S,\pi) = \alpha(w(S,\pi))$ and $(w + w')(S,\pi) = w(S,\pi) + w'(S,\pi)$ for all $(S,\pi) \in \mathcal{E}(N)$. For $N \subseteq \mathbf{U}$ and $(T,\tau) \in \mathcal{E}(N)$, $T \neq \emptyset$, the **Dirac (TUX) game** $\delta_{T,\tau} \in \mathbb{W}(N)$ is given by

$$\delta_{T,\tau}(S,\pi) := \begin{cases} 1, & \text{if } (S,\pi) = (T,\tau), \\ 0, & \text{else} \end{cases} \qquad \text{for all } (S,\pi) \in \mathcal{E}(N). \tag{A.1}$$

Every TUX game $w \in \mathbb{W}(N)$, $N \subseteq \mathbf{U}$ has a unique linear representation in terms of Dirac games,

$$w = \sum_{(T,\tau)\in\mathcal{E}(N):T\neq\emptyset} w(T,\tau)\,\delta_{T,\tau}. \tag{A.2}$$

That is, the set $\{\delta_{T,\tau} \mid (T,\tau) \in \mathcal{E}(N) : T \neq \emptyset\}$ is as basis of the vector space $\mathbb{W}(N)$.

For $N \subseteq \mathbf{U}$ and $T \subseteq N$, $T \neq \emptyset$, the **Dirac TU game** $\delta_T^N \in \mathbb{V}(N)$ is given by $\delta_T^N(S) = 1$ for $T = S$ and $\delta_T^N(S) = 0$ otherwise (we carry the superscript indicating the player set for such TU games). For $N \subseteq \mathbf{U}$ and $T \subseteq N$, $T \neq \emptyset$, the **unanimity TU game** $u_T^N \in \mathbb{V}(N)$ is given by $u_T^N(S) = 1$ for $T \subseteq S$ and $u_T^N(S) = 0$ otherwise.

Appendix B. Proofs

We now present the proofs of our results.

Appendix B.1. Proof of Proposition 2

Both, the mappings Pot and E_p are linear. Hence, it suffices to show the claim for Dirac TU games.

(i) $\Rightarrow$ (ii) From (7), we obtain the potential for the Dirac TU games, δ_T^N, $T \subseteq N$, $T \neq \emptyset$, $N \subseteq \mathbf{U}$,

$$\mathrm{Pot}\left(\delta_T^N\right) = \frac{(n-t)!\,(t-1)!}{n!}. \tag{B.1}$$

By **GEN**, we further have $\mathrm{Pot}\left(\delta_T^N\right) = \mathrm{E}_p\left(\delta_T^N\right)$. From the definition of E_p, we get

$$\mathrm{E}_p\left(\delta_T^N\right) = \sum_{\pi\in\Pi(N):\pi\ni T} p_N(\pi).$$

This confirms the claim.

(ii) $\Rightarrow$ (iii) The RHS of (19) can be written as $\frac{n}{n-s}\sum_{\pi\in\Pi(N\setminus S)} p_N(\{S\}\cup\pi)$. The claim now follows from (ii).

(iii) $\Rightarrow$ (i) We proceed by induction on n. *Induction basis:* For $n = 1$, $\sum_{\pi\in\Pi(N)} p_N(\pi) = 1$ implies the claim. *Induction hypothesis (IH):* For all $N \subseteq \mathbf{U}$, $v \in \mathbb{V}(N)$ such that $n < m$, we have $\mathrm{E}_p(v) = \mathrm{Pot}(v)$. *Induction step:* Now let $N \subseteq \mathbf{U}$ be such that $n = m$. For $T \subsetneq N$,

$i \in N \setminus T$ we have

$$\begin{aligned}
\mathrm{E}_p\left(\delta_T^N\right) &\overset{(17)}{=} \sum_{\pi\in\Pi(N\setminus T)} p_N\left(\{T\}\cup\pi\right) \\
&\overset{(\text{iii})}{=} \frac{n-t}{n}\cdot \sum_{\tau\in\Pi((N\setminus\{i\})\setminus T)} p_{N\setminus\{i\}}\left(\{T\}\cup\tau\right) \\
&\overset{(17)}{=} \frac{n-t}{n}\cdot \mathrm{E}_p\left(\delta_T^{N\setminus\{i\}}\right) \\
&\overset{IH}{=} \frac{n-t}{n}\cdot \mathrm{Pot}\left(\delta_T^{N\setminus\{i\}}\right) \\
&\overset{(\text{B.1})}{=} \frac{n-t}{n}\cdot \frac{(n-t-1)!\,(t-1)!}{(n-1)!} \\
&\overset{(\text{B.1})}{=} \mathrm{Pot}\left(\delta_T^N\right),
\end{aligned}$$

where we use the definition (17) of E_p in the first and third equation, condition (iii) in the second equation, the induction hypothesis in the fourth equation, and the formula for the potential of TU Dirac games (B.1) in the last equation.

Fix some player $i \in N$. Summing up $\sum_{T\subsetneq N:i\in T} \mathrm{Pot}\left(\delta_T^N\right) = \sum_{\pi\in\Pi(N\setminus T)} p_N\left(\{T\}\cup\pi\right)$ for all $T \subsetneq N$ and such $i \in T$ gives

$$\sum_{T\subsetneq N:i\in T} \mathrm{Pot}\left(\delta_T^N\right) = \sum_{T\subsetneq N:i\in T}\ \sum_{\tau\in\Pi(N\setminus T)} p_N\left(\{T\}\cup\tau\right) = \sum_{\pi\in\Pi(N)} p_N(\pi) - p_N\left(\{N\}\right).$$

Note that $\sum_{T\subsetneq N:i\in T} \mathrm{Pot}\left(\delta_T^N\right) = \mathrm{Pot}\left(u_{\{i\}}^N - \delta_N^N\right) = 1-\frac{1}{n}$. On the other hand, we have $\sum_{\pi\in\Pi(N)} p_N(\pi) - p_N(\{N\}) = 1 - p_N(\{N\})$. Hence, $p_N(\{N\}) = \frac{1}{n}$, and therefore $\mathrm{E}_p\left(\delta_N^N\right) = p_N(\{N\}) = \mathrm{Pot}\left(\delta_N^N\right)$.

Thus, we have $\mathrm{E}_p\left(\delta_T^N\right) = \mathrm{Pot}\left(\delta_T^N\right)$ for all $T \subseteq N$. Note that the last equation is linear in the game $v = \sum_{T\neq\emptyset} v(T)\,\delta_T^N$, so that

$$\mathrm{E}_p(v) = \sum_{T\subseteq N:T\neq\emptyset} v(T)\,\mathrm{E}_p\left(\delta_T^N\right) = \sum_{T\subseteq N:T\neq\emptyset} v(T)\,\mathrm{Pot}\left(\delta_T^N\right) = \mathrm{Pot}(v)$$

for all $v \in \mathbb{V}(N)$.

Appendix B.2. Proof of Corollary 5

(i) This immediately follows from setting $\pi = \{N\}$ and $\pi = \{N\setminus\{i\},\{i\}\}$ in (18), respectively.

(ii) This immediately follows from (i) and p_N being a probability distribution on $\Pi(N)$.

Appendix B.3. Generalization of the example with four players in (20) to larger games

There exist random partitions that satisfy **GEN** but deviate from $p^\star$ for arbitrary $n \geq 4$. Let $\mathbf{U}_{>3} := \{4, 5, 6, \dots\}$ and $\varepsilon : \mathbf{U}_{>3} \to \mathbb{R}$, $k \mapsto \varepsilon_k$ be such that $\varepsilon_k \in \left[-\frac{1}{k!}, \frac{\binom{k}{2}}{2 \cdot k!}\right]$. Consider the random partitions p^ε given by

$$p_N^\varepsilon(\pi) = \begin{cases} p_N^\star(\pi) + \frac{2}{\binom{n-2}{2}\binom{n}{2}}\varepsilon_n, & n > 3,\ \pi = \{\{i,j\}, \{k,\ell\}\} \cup [N \setminus \{i,j,k,\ell\}], \\ p_N^\star(\pi) - \frac{2}{\binom{n}{2}}\varepsilon_n, & n > 3,\ \pi = \{\{i,j\}\} \cup [N \setminus \{i,j\}], \\ p_N^\star(\pi) + \varepsilon_n, & n > 3,\ \pi = [N], \\ p_N^\star(\pi), & \text{otherwise} \end{cases} \tag{B.2}$$

for all $N \subseteq \mathbf{U}$, pairwise different $i, j, k, \ell \in N$, and $\pi \in \Pi(N)$. Straightforward but tedious calculations show that these random partitions satisfy **GEN**. Details can obtained from the authors upon request.

Appendix B.4. Proof of Theorem 7

Let p be a positive random partition (**POS**) that satisfies **GEN**.

Existence: The restriction operator r^p given by (21) obviously satisfies **PNG**. We next investigate the r^p-restriction of TUX Dirac games. For all $N \subseteq \mathbf{U}$, $i \in N$, $(T, \tau) \in \mathcal{E}(N)$, $T \neq \emptyset$, $\alpha \in \mathbb{R}$, and $(S, \pi) \in \mathcal{E}(N \setminus \{i\})$, we obtain

$$\begin{aligned}
(\alpha\delta_{T,\tau})^r_{-i}(S,\pi) &\overset{(21)}{=} \frac{n}{n-s} \sum_{B \in \pi \cup \{\emptyset\}} \frac{p_N(\{S\} \cup \pi_{+i \rightsquigarrow B})}{p_{N \setminus \{i\}}(\{S\} \cup \pi)} \alpha\delta_{T,\tau}(S, \pi_{+i \rightsquigarrow B}) \\
&\overset{(A.1)}{=} \sum_{B \in \pi \cup \{\emptyset\}} \begin{cases} \alpha \frac{n}{n-s} \frac{p_N(\{S\} \cup \pi_{+i \rightsquigarrow B})}{p_{N \setminus \{i\}}(\{S\} \cup \pi)}, & \text{if } (S, \pi_{+i \rightsquigarrow B}) = (T, \tau), \\ 0, & \text{if } (S, \pi_{+i \rightsquigarrow B}) \neq (T, \tau) \end{cases} \\
&= \begin{cases} \alpha \frac{n}{n-t} \frac{p_N(\{T\} \cup \tau)}{p_{N \setminus \{i\}}(\{T\} \cup \tau_{-i})}, & \text{if } i \in N \setminus T \text{ and } (S, \pi) = (T, \tau_{-i}), \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

where we use the definition of the restriction operator r^p (21) in the first equation and the definition of TUX Dirac games (A.1) in the second equation. Therefore, we have

$$(\alpha\delta_{T,\tau})^{r^p}_{-i} \overset{(A.1)}{=} \begin{cases} \alpha \frac{n}{n-t} \frac{p_N(\{T\} \cup \tau)}{p_{N \setminus \{i\}}(\{T\} \cup \tau_{-i})} \delta_{T,\tau_{-i}}, & \text{if } i \in N \setminus T, \\ \mathbf{0}^{N \setminus \{i\}}, & \text{if } i \in T. \end{cases} \tag{B.3}$$

Next, we show that r^p is path independent (**PI**). By the linearity of r^p and since the Dirac games form a basis of the $\mathbb{W}(N)$ (see (A.2)), it suffices to show path independence for the Dirac games $\delta_{T,\tau}$. We obtain

$$\left(\left(\alpha\delta_{T,\tau}\right)_{-i}^{r^p}\right)_{-j}^{r^p} \overset{\text{(B.3)}}{=} \begin{cases} \alpha\frac{n}{n-t}\frac{n-1}{n-t-1}\frac{p_N(\{T\}\cup\tau)}{p_{N\setminus\{i.j\}}\left(T\cup\tau_{-\{i,j\}}\right)}\delta_{T,\tau_{-\{i,j\}}}, & \text{if } i,j\in N\setminus T, \\ \mathbf{0}^{N\setminus\{i,j\}}, & \text{if } i\in T \text{ or } j\in T \end{cases}$$
$$\overset{\text{(B.3)}}{=} \left(\left(\delta_{T,\tau}\right)_{-j}^{r^p}\right)_{-i}^{r^p}$$

for all $N\subseteq\mathbf{U}$, $(T,\tau)\in\mathcal{E}(N)$, and $i,j\in N$, $i\neq j$, which establishes **PI**. We can therefore remove multiple players in a well-defined manner. In particular, we have

$$\left(\delta_{T,\tau}\right)_{-S}^{r^p} \overset{\text{(B.3)}}{=} \prod_{k=0}^{s-1}\frac{n-k}{n-t-k}\frac{p_N(\{T\}\cup\tau)}{p_{N\setminus S}(\{T\}\cup\tau_{-S})}\delta_{T,\tau_{-S}} \tag{B.4}$$

for all $N\in\mathcal{N}$, $(T,\tau)\in\mathcal{E}(N)$, and $S\in N\setminus T$.

Finally, we show $\mathrm{Pot}^{r^p}=\mathrm{E}_p$. By **PI** and Theorem (3) due to Dutta et al. (2010), there exists a unique r^p-potential Pot^{r^p}. Since r^p is linear in the game and by definition of v_w^r (13) as well as the definition of the r-Potential (15), the r^p-potential Pot^{r^p} is linear in the game. Moreover, the mapping E_p is linear in the game. Since TUX Dirac games constitute a basis, (A.2), it suffices to show the claim for TUX Dirac games. For $N\subseteq\mathbf{U}$ and $(T,\tau)\in\mathcal{E}(N)$, $T\neq\emptyset$, it follows with the definition of the auxiliary game v_w^r (13) that

$$v_{\delta_{T,\tau}}^{r^p} \overset{(13),(\text{B.4})}{=} \prod_{k=0}^{n-t-1}\frac{n-k}{n-t-k}\frac{p_N(\{T\}\cup\tau)}{p_T(\{T\})}\delta_T^N \overset{\text{Prop. 4 (i)}}{=} \frac{n!t}{(n-t)!t!}p_N(\{T\}\cup\tau)\,\delta_T^N, \tag{B.5}$$

where we use Corollary 5 (i), i.e., $p_T(\{T\})=1/t$, in the last equation. Now, we obtain

$$\mathrm{Pot}^{r^p}\left(\delta_{T,\tau}\right) \overset{(15)}{=} \mathrm{Pot}\left(v_{\delta_{T,\tau}}^{r^p}\right) \overset{(\text{B.5})}{=} \frac{n!t}{(n-t)!t!}p_N(\{T\}\cup\tau)\,\mathrm{Pot}\left(\delta_T^N\right)$$
$$\overset{(\text{B.1})}{=} p_N(\{T\}\cup\tau) \overset{(17)}{=} \mathrm{E}_p\left(\delta_{T,\tau}\right),$$

where we use the definition of the r-Potential (15) in the first equation, linearity of the potential for TU games (B.5) in the second equation, the formula for the potential of Dirac TU games (B.1) in the third equation, and the definition of E_p (17) in the fourth equation. This concludes the proof of existence.

Uniqueness: Let r be a restriction operator that satisfies **PI**, **PNG**, and (†) $\mathrm{Pot}^r=\mathrm{E}_p$. We show $r=r^p$ by induction on n.

Induction basis: For $w\in\mathbb{W}(N)$, $N\subseteq\mathbf{U}$, $n=1$, we have $w_{-i}^r=0=w_{-i}^{r^p}$ for $i\in N$, where $\{0\}=\mathbb{W}(\emptyset)$.

Induction hypothesis (IH): Suppose we have $w_{-i}^r=w_{-i}^{r^p}$ for all $w\in\mathbb{W}(N)$, $N\subseteq\mathbf{U}$ and $i\in N$ such that $n\leq m$.

Induction step: Let $N\subseteq\mathbf{U}$ be such that $n=m+1$. For all $w\in\mathbb{W}(N)$, $i\in N$, and

$(T,\tau) \in \mathcal{E}(N \setminus \{i\})$, $T \neq \emptyset$, let $w_{i,T,\tau} \in \mathbb{W}(N)$ be given by

$$w_{i,T,\tau} \equiv \sum_{B \in \tau \cup \{\emptyset\}} w\left(T, \tau_{+i \rightsquigarrow B}\right) \delta_{T,\tau_{+i \rightsquigarrow B}}. \tag{B.6}$$

Note that in the game $w_{i,T,\tau}$ those embedded coalitions for N that determine $w^r_{-i}(T,\tau)$ according to the definition of a restriction operator (**RES**) generate the same worth as in w, whereas all other embedded coalition generate a zero worth. This is what drives the results in the next two paragraphs—in the restricted game, only a few embedded coalitions generate a worth that differs from their worth in the restriction of the null game, which by **PNG** is the null game.

We have

$$\left(w_{i,T,\tau}\right)^r_{-i} \overset{\text{(B.6)},\textbf{RES}}{=} \xi\left(w_{i,T,\tau}, i\right) \delta_{T,\tau} + \left(\mathbf{0}^N\right)^r_{-i} \overset{\textbf{PNG}}{=} \xi\left(w_{i,T,\tau}, i\right) \delta_{T,\tau} \tag{B.7}$$

for some $\xi\left(w_{i,T,\tau}, i\right) \in \mathbb{R}$; and further using the definition of TUX Dirac games (A.1), we have

$$\left(w_{i,T,\tau}\right)^r_{-i}(T,\tau) \overset{\text{(A.1)}}{=} \xi\left(w_{i,T,\tau}, i\right). \tag{B.8}$$

For $k \in T$, we have

$$\left(w_{i,T,\tau}\right)^r_{-k} \overset{\text{(B.6)},\textbf{RES}}{=} \left(\mathbf{0}^N\right)^r_{-k} \overset{\textbf{PNG}}{=} \mathbf{0}^{N\setminus\{k\}}. \tag{B.9}$$

For $k \in (N \setminus \{i\}) \setminus T$, we have

$$\begin{aligned}\left(w_{i,T,\tau}\right)^r_{-k} &\overset{\text{(B.6)},\textbf{RES}}{=} \sum_{B \in \tau_{-k} \cup \{\emptyset\}} \xi\left(w_{i,T,\tau}, B, k\right) \delta_{T,\left(\tau_{-k}\right)_{+i \rightsquigarrow B}} + \left(\mathbf{0}^N\right)^r_{-k} \\ &\overset{\textbf{PNG}}{=} \sum_{B \in \tau_{-k} \cup \{\emptyset\}} \xi\left(w_{i,T,\tau}, B, k\right) \delta_{T,\left(\tau_{-k}\right)_{+i \rightsquigarrow B}}\end{aligned} \tag{B.10}$$

for some $\xi\left(w_{i,T,\tau}, B, k\right) \in \mathbb{R}$. Hence, we obtain

$$\begin{aligned}\left(\left(w_{i,T,\tau}\right)^r_{-i}\right)^r_{-k} &\overset{IH}{=} \left(\left(w_{i,T,\tau}\right)^r_{-i}\right)^{r^p}_{-k} \\ &\overset{\text{(B.7)}}{=} \left(\xi\left(w_{i,T,\tau}, i\right) \delta_{T,\tau}\right)^{r^p}_{-k} \\ &\overset{\text{(21)},\text{(B.3)}}{=} \xi\left(w_{i,T,\tau}, i\right) \frac{n-1}{n-1-t} \frac{p_{N\setminus\{i\}}\left(\{T\} \cup \tau\right)}{p_{N\setminus\{i,k\}}\left(\{T\} \cup \tau_{-k}\right)} \delta_{T,\tau_{-k}},\end{aligned} \tag{B.11}$$

where we use linearity following from the definition of the r^p restriction operator (21) and the restriction of TUX Dirac games (B.3) in the third equation. Analogously, we have

$$\begin{aligned}
&\left(\left(w_{i,T,\tau}\right)^{r}_{-k}\right)^{r}_{-i}\\
&\overset{IH}{=} \left(\left(w_{i,T,\tau}\right)^{r}_{-k}\right)^{r^p}_{-i}\\
&\overset{(B.10)}{=} \left(\sum_{B\in\tau_{-k}\cup\{\emptyset\}} \xi\left(w_{i,T,\tau},B,k\right)\delta_{T,\left(\tau_{-k}\right)_{+i\rightsquigarrow B}}\right)^{r^p}_{-i}\\
&\overset{(21),(B.3)}{=} \sum_{B\in\tau_{-k}\cup\{\emptyset\}} \xi\left(w_{i,T,\tau},B,k\right)\frac{n}{n-t}\frac{p_{(N\setminus\{k\})}\left(\{T\}\cup\left(\tau_{-k}\right)_{+i\rightsquigarrow B}\right)}{p_{N\setminus\{i,k\}}\left(\{T\}\cup\left(\left(\tau_{-k}\right)_{+i\rightsquigarrow B}\right)_{-i}\right)}\delta_{T,\left(\left(\tau_{-k}\right)_{+i\rightsquigarrow B}\right)_{-i}}\\
&= \left(\sum_{B\in\tau_{-k}\cup\{\emptyset\}} \frac{n-1}{n-1-t}\frac{p_{(N\setminus\{k\})}\left(\{T\}\cup\left(\tau_{-k}\right)_{+i\rightsquigarrow B}\right)}{p_{N\setminus\{i,k\}}\left(\{T\}\cup\tau_{-k}\right)}\xi\left(w_{i,T,\tau},B,k\right)\right)\delta_{T,\tau_{-k}}.
\end{aligned}\tag{B.12}$$

Since r satisfies **PI**, the left-hand sides of (B.11) and (B.12) coincide. We obtain

$$\begin{aligned}
&p_{N\setminus\{i\}}\left(\{T\}\cup\tau\right)\xi\left(w_{i,T,\tau},i\right)\\
&\overset{(B.11),(B.12)}{=} \left(\sum_{B\in\tau_{-k}\cup\{\emptyset\}} p_{(N\setminus\{k\})}\left(\{T\}\cup\left(\tau_{-k}\right)_{+i\rightsquigarrow B}\right)\right)\xi\left(w_{i,T,\tau},B,k\right)
\end{aligned}\tag{B.13}$$

for all $k\in(N\setminus\{i\})\setminus T$.

Further, we obtain

$$\begin{aligned}
0 &\overset{(B.6)}{=} w_{i,T,\tau}\left(N,\emptyset\right)\\
&\overset{(12),(\dagger)}{=} \sum_{\ell\in N}\left[\mathrm{E}_p\left(w_{i,T,\tau}\right)-\mathrm{E}_p\left(\left(w_{i,T,\tau}\right)^{r}_{-\ell}\right)\right]\\
&\overset{(B.7),(B.10),(B.9)}{=} n\mathrm{E}_p\left(w_{i,T,\tau}\right)-\mathrm{E}_p\left(\xi\left(w_{i,T,\tau},i\right)\delta_{T,\tau}\right)-\sum_{\ell\in T}\mathrm{E}_p\left(\mathbf{0}^{N\setminus\{l\}}\right)\\
&\qquad -\sum_{\ell\in(N\setminus\{i\})\setminus T}\mathrm{E}_p\left(\sum_{B\in\tau_{-\ell}\cup\{\emptyset\}}\xi\left(w_{i,T,\tau},B,\ell\right)\delta_{T,\left(\tau_{-\ell}\right)_{+i\rightsquigarrow B}}\right)\\
&\overset{(17),(A.1)}{=} n\mathrm{E}_p\left(w_{i,T,\tau}\right)-\xi\left(w_{i,T,\tau},i\right)p_{N\setminus\{i\}}\left(\tau\cup\{T\}\right)\\
&\qquad -\sum_{\ell\in(N\setminus\{i\})\setminus T}\ \sum_{B\in\tau_{-\ell}\cup\{\emptyset\}}\xi\left(w_{i,T,\tau},B,\ell\right)p_{N\setminus\{\ell\}}\left(\left(\tau_{-\ell}\right)_{+i\rightsquigarrow B}\cup\{T\}\right)\\
&\overset{(B.13)}{=} n\mathrm{E}_p\left(w_{i,T,\tau}\right)-(n-t)\,p_{N\setminus\{i\}}\left(\{T\}\cup\tau\right)\xi\left(w_{i,T,\tau},i\right),
\end{aligned}\tag{B.14}$$

where in the second equation we use the formula for the potential (12) and the fact that (†)

$\mathrm{Pot}^r = \mathrm{E}_p$; in the the fourth equation refer to the definition of E_p (17) and the definition of TUX dirac games (A.1). Finally, we obtain

$$\begin{aligned}
w^r_{-i}(T,\tau) &\overset{\mathbf{R},(B.6)}{=} (w_{i,T,\tau})^r_{-i}(T,\tau) \\
&\overset{(B.8)}{=} \xi(w_{i,T,\tau}, i) \\
&\overset{(B.14)}{=} \frac{n}{n-t}\frac{1}{p_{N\setminus\{i\}}(\{T\}\cup\tau)}\mathrm{E}_p(w_{i,T,\tau}) \\
&\overset{(17),(B.6),(A.1)}{=} \frac{n}{n-t}\sum_{B\in\tau\cup\{\emptyset\}} w(T,\tau_{+i\rightsquigarrow B})\frac{p_N(\{T\}\cup\tau_{+i\rightsquigarrow B})}{p_{N\setminus\{i\}}(\{T\}\cup\tau)} \\
&\overset{(21)}{=} w^{r^p}_{-i}(T,\tau),
\end{aligned}$$

where in the fourth equation we refer to the definition of E_p (17) and the definition of TUX Dirac games (A.1); and in the fifth equation we refer to the definition of the r^p-restriction operator (21). This concludes the proof.

Appendix B.5. Proof of Corollary 8

(i) Note that all mappings involved are linear. Since the Dirac games form a basis of the $\mathbb{W}(N)$ (see (A.2)), it suffices to establish the formula for Dirac games. For all $N\subseteq\mathbf{U}$, $i\in N$, and $(T,\tau)\in\mathcal{E}(N)$, $T\neq\emptyset$, we obtain

$$\mathrm{Sh}^{r^p}_i(\delta_{T,\tau}) \overset{(14)}{=} \mathrm{Sh}_i\left(v^{r^p}_{\delta_{T,\tau}}\right) \overset{(B.5),(1)}{=} \begin{cases} p_N(\{T\}\cup\tau), & i\in T, \\ -\frac{t}{n-t}p_N(\{T\}\cup\tau), & i\in N\setminus T \end{cases}, \tag{B.15}$$

where in the first equation is due to the definition of the r-Shapley value (14) and the second equation is due to $v^{r^p}_{\delta_{T,\tau}} = \frac{n!t}{(n-t)!t!}p_N(\{T\}\cup\tau)\,\delta^N_T$ and the Shapley value for TU Dirac games. This verifies the formula for Dirac games.

(ii) Since both solutions are linear, it suffices to show the claim for Dirac TU games. For all $N\subseteq\mathbf{U}$, $T\subseteq N$, $T\neq\emptyset$, and $i\in T$, we obtain

$$\begin{aligned}
\mathrm{Sh}^{r^p}_i\left(\delta^N_T\right) = \mathrm{Sh}^{r^p}_i\left(\sum_{\tau\in\Pi(N\setminus T)}\delta^N_{T,\tau}\right) &\overset{(B.15)}{=} \sum_{\tau\in\Pi(N\setminus T)} p_N(\{T\}\cup\tau) \\
&\overset{\text{Prop 4 (ii)}}{=} \frac{(n-t)!\,(t-1)!}{n!} \overset{(1)}{=} \mathrm{Sh}_i\left(\delta^N_T\right),
\end{aligned}$$

where the second equation follows from Proposition 4 (ii) and the last equation follows from the definition of the Shapley value (1). Analogously for $i\in N\setminus T$.

Appendix B.6. Proof of Lemma 9

By (5), we have

$$\frac{1}{n-s}=\frac{n}{n-s}\frac{p^\star_N\left(\{S\}\cup\pi_{+i\rightsquigarrow\emptyset}\right)}{p^\star_{N\setminus\{i\}}\left(\{S\}\cup\pi\right)}\quad\text{and}\quad\frac{b}{n-s}=\frac{n}{n-s}\frac{p^\star_N\left(\{S\}\cup\pi_{+i\rightsquigarrow B}\right)}{p^\star_{N\setminus\{i\}}\left(\{S\}\cup\pi\right)}\quad\text{for } B\in\pi$$

for all $N\subseteq\mathbf{U}$, $i\in N$, and $(S,\pi)\in\mathcal{E}\left(N\setminus\{i\}\right)$. The random partition $p^\star$ is positive (**POS**) and generates the potential for TU games (**GEN**). Hence, Theorem 7 applies. The first claim now drops from (23) and (21).

Since both MPW and $\mathrm{Sh}^{r^\star}$ are linear, it suffices to show the second claim for Dirac games. For all $N\subseteq\mathbf{U}$, $T\subseteq N$, and $i\in N$, we have

$$\begin{aligned}\mathrm{MPW}_i\left(\delta_{T,\tau}\right)&\overset{(10)}{=}\mathrm{Sh}_i\left(\bar{v}^\star_{\delta_{T,\tau}}\right)\\&\overset{(11)}{=}\mathrm{Sh}_i\left(p^\star_{N\setminus T}\left(\tau\right)\delta^N_T\right)\\&\overset{(1)}{=}\begin{cases}\frac{p^\star_{N\setminus T}(\tau)}{n\binom{n-1}{t-1}}, & i\in T,\\ -\frac{t}{n-t}\frac{p^\star_{N\setminus T}(\tau)}{n\binom{n-1}{t-1}}, & i\in N\setminus T\end{cases}\\&\overset{(5)}{=}\begin{cases}p^\star_N\left(\{T\}\cup\tau\right), & i\in T,\\ -\frac{t}{n-t}p^\star_N\left(\{T\}\cup\tau\right), & i\in N\setminus T,\end{cases}\end{aligned}$$

where the first equation is due to the definition of MPW (10), the second equation is due to the definition of the average game $\bar{v}^\star$ (11), the third equation is due to the definition of the Shapley value (1), and the fourth equation is due to the definition of $p^\star$ (5). The claim now follows from (B.15).

Appendix B.7. Proof of Theorem 10

By definition of the random partition $p^\star$ (5), $p^\star$ is positive (**POS**). For all $N\subseteq\mathbf{U}$, $w\in\mathbb{W}\left(N\right)$, $i\in N$, and $(T,\tau)\in\mathcal{E}\left(N\setminus\{i\}\right)$, $T\neq\emptyset$, we have

$$p^\star_N\left(\{T\cup\{i\}\}\cup\tau\right)\overset{(5)}{=}\frac{t}{n-t}\sum_{B\in\tau\cup\{\emptyset\}}p^\star_N\left(\{T\}\cup\tau_{+i\rightsquigarrow B}\right). \tag{B.16}$$

Hence, we obtain

$$\mathrm{Sh}^{r^{p^\star}}_i\left(w\right)=\sum_{(T,\tau)\in\mathcal{E}(N\setminus\{i\})}\left(\frac{t}{n-t}\sum_{B\in\tau\cup\{\emptyset\}}p^\star_N\left(\{T\}\cup\tau_{+i\rightsquigarrow B}\right)\left[w\left(T\cup\{i\},\tau\right)-w\left(T,\tau_{+i\rightsquigarrow B}\right)\right]\right)$$

for all $N\subseteq\mathbf{U}$, $w\in\mathbb{W}\left(N\right)$, and $i\in N$. This already entails that $\mathrm{Sh}^{r^{p^\star}}$ satisfies **NP**.

Let now p be a positive random partition (**POS**). If $n\leq1$, then trivially $p_N=p^\star_N$. Let now $N\subseteq\mathbf{U}$ be such that $n>1$. For $\alpha\in\mathbb{R}$, $i\in N$, $\pi\in\Pi\left(N\setminus\{i\}\right)$, and $B\in\Pi$, let

$w^{\alpha}_{i,\pi,B} \in \mathbb{W}(N)$ be given by

$$w^{\alpha}_{i,\pi,B} \equiv \alpha \delta_{B \cup \{i\}, \pi \setminus \{B\}} + \sum_{C \in (\pi \setminus \{B\}) \cup \{\emptyset\}} \alpha \delta_{B, (\pi \setminus \{B\})_{+i \rightsquigarrow C}}.$$

By construction, player i is a null player in $w^{\alpha}_{i,\pi,B}$ for any $\alpha \in \mathbb{R}$. By **NP**, we therefore have $\mathrm{Sh}^{rp}_i \left(w^1_{i,\pi,B} \right) = 0$.By (22), we have

$$\mathrm{Sh}^{rp}_i \left(w^{\alpha}_{i,\pi,B} \right) = \alpha \left(p_N \left(\pi_{+i \rightsquigarrow B} \right) - \frac{b}{n-b} \sum_{C \in (\pi \setminus \{\pi(i)\}) \cup \{\emptyset\}} p_N \left(\pi_{+i \rightsquigarrow C} \right) \right).$$

Hence, **NP** implies

$$p_N \left(\pi_{+i \rightsquigarrow B} \right) - \frac{b}{n-b} \sum_{C \in (\pi \setminus \{B\}) \cup \{\emptyset\}} p_N \left(\pi_{+i \rightsquigarrow C} \right) = 0. \tag{B.17}$$

By (B.16), the random partition $p^{\star}$ satisfies (B.17).

Let $\beta \in \mathbb{R}$ be defined by $p_N([N]) = \beta p^{\star}_N([N]) = \frac{\beta}{n!}$. We show $p_N(\pi) = \beta \cdot p^{\star}_N(\pi)$ for all $\pi \in \Pi(N)$ by induction on $|\pi|$. Since both p_N and $p^{\star}_N$ are probability distributions on $\Pi(N)$, this implies $p_N = p^{\star}_N$.

Induction basis: By construction, the claim holds for $|\pi| = n$, that is, $\pi = [N]$.

Induction hypothesis (IH): Let the claim hold for all $\pi \in \Pi(N)$ with $|\pi| \geq t > 1$.

Induction step: Fix $i \in N$. Let now $\pi \in \Pi(N \setminus \{i\})$ be such that $|\pi| = t - 1$. By (B.17), we have

$$p_N \left(\pi_{+i \rightsquigarrow B} \right) - \frac{b}{n-b} \sum_{C \in \pi \setminus \{B\}} p_N \left(\pi_{+i \rightsquigarrow C} \right) = \frac{b}{n-b} p_N \left(\pi_{+i \rightsquigarrow \emptyset} \right) \overset{IH}{=} \frac{b \cdot \beta}{n-b} p^{\star}_N \left(\pi_{+i \rightsquigarrow \emptyset} \right) \tag{B.18}$$

for all $B \in \pi$. That is, we have a system of $|\pi|$ linear equations indexed by $B \in \pi$ with $|\pi|$ unknowns, $p_N(\pi_{+i \rightsquigarrow C})$, indexed by $C \in \pi$. This system be written as $Ax = y$. The column vector $x \equiv (p_N(\pi_{+i \rightsquigarrow C}))_{C \in \pi}$ represents the $|\pi|$ unknowns. The column vector $y \equiv \left(\frac{b \cdot \beta}{n-b} p^{\star}_N \left(\pi_{+i \rightsquigarrow \emptyset} \right) \right)_{B \in \pi}$ represents the right-hand side of (B.18). The $|\pi| \times |\pi|$ matrix $A \equiv (a_{B,C})_{B,C \in \pi}$ given by

$$a_{B,C} \equiv \begin{cases} 1, & C = B, \\ -\frac{b}{n-b}, & C \neq B \end{cases}$$

represents the coefficients of (B.18), where the left index indicates the equation and the right index indicates the unknown. Note that we cover all $\pi \in \Pi(N)$ with $|\pi| = t - 1$.

By (B.16), the column vector $x^{\star} \equiv (\beta \cdot p^{\star}_N(\pi_{+i \rightsquigarrow C}))_{C \in \pi}$ is a solution of (B.18). Remains to show that $x^{\star}$ is the unique solution. This is the case if and only if A is non-singular.

Consider the $|\pi| \times |\pi|$ matrix $D = (d_{B,C})_{B,C\in\pi}$ given by

$$d_{B,C} \equiv \begin{cases} p_N^\star(\pi_{+i\rightsquigarrow B}), & C = B, \\ 0, & C \neq B. \end{cases}$$

This matrix is non-singular. Hence, the matrix A is non-singular if and only if the product $AD = (e_{B,C})_{B,C\in\pi}$ given by

$$e_{B,C} \equiv \begin{cases} p_N^\star(\pi_{+i\rightsquigarrow B}), & C = B, \\ -\frac{b}{n-b} p_N^\star(\pi_{+i\rightsquigarrow C}), & C \neq B \end{cases} \tag{B.19}$$

is non-singular. For any $B \in \pi$, we have

$$|e_{B,B}| - \sum_{C\in\pi\setminus\{B\}} |e_{B,C}| \overset{(B.19)}{=} p_N^\star(\pi_{+i\rightsquigarrow B}) - \frac{b}{n-b} \sum_{C\in\pi\setminus\{B\}} p_N^\star(\pi_{+i\rightsquigarrow C}) \overset{(B.17)}{=} \frac{b}{n-b} p_N^\star(\pi_{+i\rightsquigarrow\emptyset}) \overset{(5)}{>} 0,$$

where the last inequality is follows from the definition of $p^\star$. Hence, the matrix AD has a strictly dominant diagonal. It is well-known that real square matrices with a strictly dominant diagonal are non-singular (see for example Taussky, 1949, Theorem 1), which concludes the proof.

Appendix B.8. Proof of Proposition 13

We prepare the proof by a lemma.

Lemma 15. *For all $w \in \mathbb{W}(N)$, $N \subseteq \mathbf{U}$ and $i \in N$, we have $\bar{v}^\star_{w^{r^\star}_{-i}} = (\bar{v}^\star_w)_{-i}$.*

Proof. The claim follows from the linearity of the involved operators, from

$$\bar{v}^\star_{(\delta_{T,\tau})^{r^\star}_{-i}} \overset{(B.3)}{=} \bar{v}^\star_{\delta_{T,\tau_{-i}}} \overset{(11)}{=} \delta_T^{N\setminus\{i\}} \overset{(11)}{=} \left(\bar{v}^\star_{\delta_{T,\tau}}\right)_{-i}$$

for all $N \subseteq \mathbf{U}$, $T \subseteq N$, $T \neq \emptyset$, $(T,\tau) \in \mathcal{E}(N)$, and $i \in N \setminus T$, and

$$\bar{v}^\star_{(\delta_{T,\tau})^{r^\star}_{-i}} \overset{(B.3)}{=} \bar{v}^\star_{\mathbf{0}^{N\setminus\{i\}}} \overset{(11)}{=} \mathbf{0}^{N\setminus\{i\}} \overset{(11)}{=} \left(\bar{v}^\star_{\delta_{T,\tau}}\right)_{-i}$$

for $i \in T$, where we first use the restriction formula for TUX Dirac games (B.3) and then definition of the average game(11). □

We can now proof the proposition.

Proof of Proposition 13. For all $w \in \mathbb{W}(N)$, $N \subseteq \mathbf{U}$ and $i, j \in N$, $i \neq j$ such that i is a strong null player in w, we have

$$\mathrm{MPW}_i\left(w^{r^\star}_{-j}\right) \overset{(10)}{=} \mathrm{Sh}_i\left(\bar{v}^\star_{w^{r^\star}_{-j}}\right) \overset{\text{Lemma 15}}{=} \mathrm{Sh}_i\left((\bar{v}^\star_w)_{-j}\right) = 0,$$

where the first equation just follows from the definition of MPW and the third equation follows from the facts that the strong null player i becomes a null player in the average game $\bar{v}_w^\star$ (Macho-Stadler et al., 2007, p. 346), that null players remain null players in a subgame of a TU game, and that null players obtain a zero Shapley payoff. $\square$

Appendix B.9. Proof of Proposition 14

Existence: By Theorem 2, the random partition $p^\star$ satisfies **GEN**. The Ewens distributions defined in Footnote 3, in particular $p^\star$, are known to satisfy **CI**.

Uniqueness: Let the random partition p for $\mathbf{U}$ satisfy **CI** and **GEN**. We show $p_N = p_N^\star$ for all $N \subseteq \mathbf{U}$ by induction on n.

Induction basis: For $n = 1$, the claim is immediate.

Induction hypothesis (IH): Suppose $p_N = p_N^\star$ for all $N \subseteq \mathbf{U}$ such that $n \leq \bar{n}$.

Induction step: Let $N \subseteq \mathbf{U}$ be such that $n = \bar{n} + 1$. We have

$$p_N(\{N\}) \overset{\mathbf{GEN}}{=} \mathrm{Pot}\left(u_N^N\right) \overset{(4)}{=} p_N^\star(\{N\}),$$

where the second equation is due to the fact that $p^\star$ generates the potential (4). Consider now $\pi \in \Pi(N)$, $\pi \neq \{N\}$ and $B \in \pi$. We obtain

$$\begin{aligned} p_N(\pi) &\overset{\mathbf{CI}}{=} p_{N\setminus B}(\pi \setminus \{B\}) \sum_{\tau \in \Pi(N): B \in \tau} p_N(\tau) \\ &\overset{\mathbf{GEN}}{=} p_{N\setminus B}(\pi \setminus \{B\}) \,\mathrm{Pot}\left(\delta_B^N\right) \\ &\overset{IH}{=} p^\star_{N\setminus B}(\pi \setminus \{B\}) \,\mathrm{Pot}\left(\delta_B^N\right) \\ &\overset{(4)}{=} p^\star_{N\setminus B}(\pi \setminus \{B\}) \sum_{\tau \in \Pi(N): B \in \tau} p_N^\star(\tau) \\ &\overset{\mathbf{CI}}{=} p_N^\star(\pi), \end{aligned}$$

which concludes the proof.

References

Albizuri, M. J., Arin, J., Rubio, J., 2005. An axiom system for a value for games in partition function form. International Game Theory Review 7, 63–73.

Aldous, D., 1985. Exchangeability and related topics. In: Hennequin, P. L. (Ed.), Ecole d'été de probabilités de Saint-Flour, XIII—1983. Vol. 1117 of Lecture Notes in Mathematics. Springer, Berlin, Ch. 1, pp. 1–198.

Bolger, E. M., 1989. A set of axioms for a value for partition function games. International Journal of Game Theory 18, 37–44.

Brügemann, B., Gautier, P., Menzio, G., 2018. Intra firm bargaining and Shapley values. The Review of Economic Studies 86 (2), 564–592.

Calvo, E., Santos, J. C., 1997. Potentials in cooperative TU games. Mathematical Social Sciences 34, 175–190.

Casajus, A., 2014. Potential, value, and random partitions. Economics Letters 126 (2), 164–166.

Casajus, A., Huettner, F., 2018. Decomposition of solutions and the Shapley value. Games and Economic Behavior 13 (3), 1–23.

Castro, J., Gómez, D., Tejada, J., 2009. Polynomial calculation of the Shapley value based on sampling. Computers and Operations Research 36 (5), 1726–1730.

de Clippel, G., Serrano, R., 2008. Marginal contributions and externalities in the value. Econometrica 76 (6), 1413–1436.

Crane, H., 2016. The ubiquitous Ewens sampling formula. Statistical Science 31 (1), 1–19.

Dutta, B., Ehlers, L., Kar, A., 2010. Externalities, potential, value and consistency. Journal of Economic Theory 145 (6), 2380–2411.

Ewens, W., 1972. The sampling theory of selectively neutral alleles. Theoretical Population Biology 3, 87–112.

Grabisch, M., Funaki, Y., 2012. A coalition formation value for games in partition function form. European Journal of Operational Research 221 (1), 175–185.

Gul, F., 1989. Bargaining foundations of Shapley value. Econometrica 57 (1), 81.

Hart, S., Mas-Colell, A., 1989. Potential, value, and consistency. Econometrica 57 (3), 589–614.

Kamijo, Y., Kongo, T., 2012. Whose deletion does not affect your payoff? The difference between the Shapley value, the egalitarian value, the solidarity value, and the Banzhaf value. European Journal of Operational Research 216 (3), 638–646.

Karlin, S., McGregor, J., 1972. Addendum to a paper of W. Ewens. Theoretical Population Biology 3 (1), 113–116.

Kingman, J. F. C., 1978. Random partitions in population genetics. Proceeedings of the Royal Society of London. A. Mathematical and Physical Sciences 361 (1704).

Kóczy, L. Á., 2018. Partition Function Form Games. Springer International Publishing.

Lundberg, S. M., Erion, G., Chen, H., DeGrave, A., Prutkin, J. M., Nair, B., Katz, R., Himmelfarb, J., Bansal, N., Lee, S.-I., 2020. From local explanations to global understanding with explainable AI for trees. Nature Machine Intelligence 2 (1), 56–67.

Macho-Stadler, I., Pérez-Castrillo, D., Wettstein, D., 2007. Sharing the surplus: An extension of the Shapley value for environments with externalities. Journal of Economic Theory 135 (1), 339–356.

McQuillin, B., 2009. The extended and generalized Shapley value: Simultaneous consideration of coalitional externalities and coalitional structure. Journal of Economic Theory 144, 696–721.

McQuillin, B., Sugden, R., 2016. Backward induction foundations of the Shapley value. Econometrica 84 (6), 2265–2280.

Michalak, T. P., Aadithya, K. V., Szczepanski, P. L., Ravindran, B., Jennings, N. R., 2013. Efficient computation of the Shapley value for game-theoretic network centrality. Journal of Artificial Intelligence Research 46, 607–650.

Myerson, R. B., 1977. Values of games in partition function form. International Journal of Game Theory 6 (1), 23–31.

Myerson, R. B., 1980. Conference structures and fair allocation rules. International Journal of Game Theory 9, 169–182.

Ortmann, K. M., 1998. Conservation of energy in value theory. Mathematical Methods of Operations Research 47, 423–449.

Pham Do, K. H., Norde, H., 2007. The Shapley value for partition function form games. International Game Theory Review 9, 353–360.

Pitman, J., 2006. Combinatorial Stochastic Processes. Springer.

Sánchez-Pérez, J., feb 2016. A decomposition for the space of games with externalities. International Journal of Game Theory 46 (1), 205–233.

Shapley, L. S., 1953. A value for n-person games. In: Kuhn, H., Tucker, A. (Eds.), Contributions to the Theory of Games. Vol. II. Princeton University Press, Princeton, pp. 307–317.

Skibski, O., Michalak, T., 2019. Fair division in the presence of externalities. International Journal of Game Theory 49 (1), 147–172.

Skibski, O., Michalak, T. P., Woolbridge, M., 2018. The stochastic Shapley value for coalitional games with

externalities. Games and Economic Bahavior 108, 65–80.
Sobolev, A. I., 1975. The characterization of optimality principles in cooperative games by functional equations. In: Vorobev, N. N. (Ed.), Mathematical Methods in the Social Sciences. Vol. 6. Academy of Sciences of the Lithuanian SSR, Vilnius, pp. 95–151, in Russian.
Stole, L. A., Zwiebel, J., Jul. 1996. Intra-firm bargaining under non-binding contracts. The Review of Economic Studies 63 (3), 375.
Taussky, O., 1949. A recurring theorem on determinants. The American Mathematical Monthly 56 (10), 672–676.
Thrall, R. M., Lucas, W. F., 1963. n-person games in partition function form. Naval Research Logistic Quarterly 10, 281–293.
Yang, G., Sun, H., Hou, D., Xu, G., 2019. Games in sequencing situations with externalities. European Journal of Operational Research 278 (2), 699–708.
Young, H. P., 1985. Monotonic solutions of cooperative games. International Journal of Game Theory 14, 65–72.